

Residual deficiency

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Abstract

We introduce a new real valued invariant for finitely presented groups called residual deficiency. Its main property is the following. Let G be a finitely presented group. If the residual deficiency of G is greater than one, then G has a finite index subgroup with deficiency greater than one. The latter property is strong. For instance, such a group is large ([5]), has positive rank gradient and positive first L^2 -Betti number, among other properties.

We also compute the residual deficiency of some well known families of presentations, prove that residual deficiency minus one is supermultiplicative with respect to finite index normal subgroups and find lower bounds for the residual deficiency of quotients.

1 Introduction

We say a group G is large if it contains a finite index subgroup which surjects onto the non-abelian free group of rank 2. Such property, introduced in [31], is invariant under finite index subgroups, finite index supergroups and prequotients. Being large is a strong property, it implies that the group in question contains a non-abelian free subgroup [29], that it is SQ -universal ([31]) (every countable group is a subgroup of a quotient of G), it has subgroups with arbitrarily large first Betti number [23], has uniformly exponential word growth [16], as well as subgroup growth of strict type n^n [24], among other properties.

In this paper, we introduce a new real valued invariant for finitely presented groups which we call *residual deficiency*. Let G be a finitely presented group, then the main property of the residual deficiency of G is that it helps construct a lower bound for the deficiency of some finite index subgroups of G . This is useful because it helps detect largeness and study the class of finitely presented large groups, as we now explain.

Deficiency and largeness were first related in [5] by B. Baumslag and S. Pride. The result says that any group with deficiency greater than one is large. Largeness is invariant under finite index supergroups. Hence, if G is a group with a finite index subgroup H which has deficiency greater than one, then G is also large. Therefore, looking at the deficiency of finite index subgroups of a group G can help detect largeness.

Having a finite index subgroup with deficiency greater than one is much stronger than being large. By recent results of I. Agol ([2]), the fundamental group of a closed, orientable, hyperbolic 3-dimensional manifold is large. By results in [13], we know that these groups, along with all its finite index subgroups have deficiency zero. This gives

a substantial collection of finitely presented groups which are large but do not have a finite index subgroup with deficiency greater than one. Therefore, the latter property is only enjoyed by some large groups and hence can be used to study the class of large finitely presented groups.

In Section 2 we introduce real valued submultiplicative and supermultiplicative invariants. A real valued invariant I , submultiplicative with respect to finite index subgroups, is one such that given a finite index subgroup H in G satisfies

$$\frac{I(H)}{|G : H|} \leq I(G). \quad (1)$$

Given an invariant I as above, C.T.C. Wall considered the following gradient ([38])

$$\tilde{I}(G) = \inf_{\substack{H \leq G \\ f}} \left\{ \frac{I(H)}{|G : H|} \right\}. \quad (2)$$

He remarked that such invariant satisfies

$$\tilde{I}(G) = \frac{\tilde{I}(H)}{|G : H|}, \quad (3)$$

where H is a finite index subgroup of G , though he did not prove it. This property characterises what he defined in [38] as generalised Euler characteristics. If the inequality in eq. (1) is inverted, we obtain a supermultiplicative invariant with respect to finite index subgroups. In this case, a gradient is defined by taking the supremum rather than the infimum.

Many examples of submultiplicative and supermultiplicative invariants can be found in the literature. For instance, consider the rank of a group G , which we denote by $d(G)$. Subtracting 1 from $d(G)$ gives a submultiplicative invariant. Its associated gradient is called the *rank gradient* which we denote by $RG(G)$. Subtracting 1 from the deficiency of a finitely presented group G is an example of a supermultiplicative invariant. We call the gradient obtained from this the *deficiency gradient* which we denote by $DG(G)$. In Section 2 we prove the equality of eq. (3), give an overview of some of these invariants and says how they relate to one another.

Section 3 starts by defining the residual deficiency of a finitely presented group. We prove its main property which is that it helps construct a lower bound for the deficiency of some finite index subgroups of the group. We then relate it to the gradients studied in Section 2 and compute the residual deficiency of some well known families of finite presentations. For example, families of presentations which define generalised triangle groups, Coxeter groups and generalised tetrahedral groups.

In Section 4 we prove that the residual deficiency of G minus 1 is a supermultiplicative invariant with respect to finite index normal subgroups. That is

$$rdef(G) - 1 \leq \frac{rdef(H) - 1}{|G : H|}, \quad (4)$$

where H is a finite index normal subgroup of G . This allows us to define the residual deficiency gradient. We show, however, that this is equal to the deficiency gradient of the group.

In Section 5, we identify conditions that give a strict inequality in eq. (4). So, G may have residual deficiency equal to one, but a strict inequality in eq. (4) implies it has a finite index subgroup with residual deficiency greater than one. This is useful as most of the properties that interest us are invariant under finite index supergroups (for example being large or satisfying $\tilde{I}(G) > 0$, where \tilde{I} is any generalised Euler characteristic). These conditions, unfortunately, do not readily apply to any group G , but do to quotients G/M where G is not residually finite.

2 Submultiplicative and supermultiplicative invariants and generalised Euler characteristics

Consider a real valued invariant I defined over a class of groups \mathfrak{L} . Let \mathfrak{L} be closed under finite index subgroups. By this we mean that if G is in \mathfrak{L} and H is a finite index subgroup of G , then H is in \mathfrak{L} . The invariant I is said to be *submultiplicative* with respect to finite index (finite index normal) subgroups if

$$\frac{I(H)}{|G : H|} \leq I(G), \quad (5)$$

where G is in \mathfrak{L} , H is a finite index (finite index normal) subgroup of G and where $|G : H|$ denotes the index of H in G .

In an analogous way, I is *supermultiplicative* with respect to finite index (finite index normal) subgroups if

$$\frac{I(H)}{|G : H|} \geq I(G).$$

In problem E10 of [38], C.T.C. Wall mentions that given a submultiplicative invariant I as above, one may define

$$\tilde{I}(G) = \inf_{\substack{H \leq G \\ f}} \left\{ \frac{I(H)}{|G : H|} \right\}. \quad (6)$$

He remarks, without giving proof, that if H is a finite index subgroup of G , then

$$\tilde{I}(G) = \frac{\tilde{I}(H)}{|G : H|}.$$

An invariant such as I which satisfies the previous equality with respect to finite index subgroups is called a *generalised Euler characteristic*.

In a similar vein, given a supermultiplicative invariant I , we define

$$\tilde{I}(G) = \sup_{\substack{H \leq G \\ f}} \left\{ \frac{I(H)}{|G : H|} \right\}.$$

Note. If I is submultiplicative only with respect to finite index normal subgroups of G , then define

$$\tilde{J}(G) = \inf_{\substack{H \trianglelefteq G \\ f}} \left\{ \frac{I(H)}{|G : H|} \right\}.$$

In the case I is supermultiplicative only with respect to finite index normal subgroups, then define

$$\tilde{J}(G) = \sup_{\substack{H \trianglelefteq G \\ f}} \left\{ \frac{I(H)}{|G : H|} \right\}.$$

Proposition 2.1 below shows that it does not matter whether we consider finite index subgroups or only finite index normal subgroups.

Proposition 2.1. *Let G be a group in a class of groups \mathfrak{L} where \mathfrak{L} is closed under finite index subgroups. Suppose I is an invariant for groups in \mathfrak{L} , submultiplicative with respect to finite index subgroups. Then*

$$\tilde{I}(G) = \tilde{J}(G). \quad (7)$$

If I is submultiplicative with respect to finite index normal subgroups, then

$$\tilde{I}(G) = \tilde{J}(G). \quad (8)$$

Proof. The set of finite index normal subgroups is contained in the set of finite index subgroups, therefore

$$\inf_{\substack{H \trianglelefteq G \\ f}} \left\{ \frac{I(H)}{|G : H|} \right\} \leq \inf_{\substack{H \trianglelefteq G \\ f}} \left\{ \frac{I(H)}{|G : H|} \right\}.$$

To prove the other inequality consider H a finite index subgroup of G . Let K be the core of H in G , defined as the intersection of all the conjugates of H in G . Note that K is a finite index normal subgroup of G . Moreover, as $K \trianglelefteq_f H$, then

$$\frac{I(K)}{|H : K|} \leq I(H).$$

This implies

$$\frac{I(K)}{|G : K|} = \frac{I(K)}{|G : H||H : K|} \leq \frac{I(H)}{|G : H|}.$$

As this holds for all finite index subgroup H of G , the result follows. \square

Note. The previous proof equally works if I is a supermultiplicative invariant. Just reverse the inequalities and change supremum for infimum in eq. (7) and eq. (8).

Proposition 2.2. *Let G be a group in a class of groups \mathfrak{L} where \mathfrak{L} is closed under finite index subgroups. Suppose I is an invariant defined over the groups in \mathfrak{L} and assume I is submultiplicative with respect to finite index normal subgroups of G . Then*

$$\tilde{I}(G) = \frac{\tilde{I}(H)}{|G : H|} \quad (9)$$

for all finite index subgroups H of G . In particular \tilde{I} is a generalised Euler characteristic.

If I is supermultiplicative with respect to finite index normal subgroups of G , then eq. (9) holds too.

Proof. Assume I is submultiplicative. The proof for when I is supermultiplicative is the same but with the inequalities inverted.

Consider K a finite index normal subgroup of G . Let $N = K \cap H$ and consider $\text{Core}_G(N)$, the core of N in G . Since $\text{Core}_G(N)$ is finite index in H , then eq. (6) and eq. (5) imply

$$\begin{aligned} \frac{\tilde{I}(H)}{|G : H|} &\leq \frac{I(\text{Core}_G(N))}{|G : H||H : \text{Core}_G(N)|} = \frac{I(\text{Core}_G(N))}{|G : \text{Core}_G(N)|} \\ &= \frac{I(\text{Core}_G(N))}{|G : K||K : \text{Core}_G(N)|} \leq \frac{I(K)}{|G : K|}. \end{aligned}$$

As this holds for all finite index normal subgroups K of G , then

$$\frac{\tilde{I}(H)}{|G : H|} \leq \tilde{I}(G).$$

By Proposition 2.1 assume \tilde{I} is defined over all finite index subgroups. If L is a finite index subgroup of H , then L is a finite index subgroup of G . Hence, eq. (6) implies

$$\tilde{I}(G) \leq \frac{I(L)}{|G : L|} = \frac{I(L)}{|G : H||H : L|}.$$

The previous inequality holds for all finite index subgroups of H , therefore

$$\tilde{I}(G) \leq \frac{\tilde{I}(H)}{|G : H|}.$$

□

Next we give examples of invariants for finitely generated and finitely presented groups which are submultiplicative or supermultiplicative. The objective is to see how they relate to each other and set the stage for a new supermultiplicative invariant which will be introduced in the following section.

The *rank* $d(G)$ of a finitely generated group G is defined as the minimum number of elements needed to generate G . Let G be a finitely generated group generated by a finite set X of size n and let H be a subgroup of finite index in G . The Nielson-Schreier method ([27], pp. 16) computes a finite generating set Y for H and shows that

$$\frac{|Y| - 1}{|G : H|} = |X| - 1.$$

As the rank of H is the minimal number of elements needed to generate the group H and the above equality holds for all generating sets X of G , then

$$\frac{d(H) - 1}{|G : H|} \leq d(G) - 1. \quad (10)$$

Therefore, the rank minus one is a submultiplicative invariant, with respect to finite index subgroups, for finitely generated groups.

In [19], M. Lackenby defined the *rank gradient* of a finitely generated group G as

$$RG(G) = \inf_f \left\{ \frac{d(H) - 1}{|G : H|} \right\},$$

where the infimum is taken over all the finite index subgroups of G . The notion was motivated by problems in 3-dimensional geometry and has received lots of attention since then.

C.T.C. Wall mentions in [38] that the rank of a group is an example of a submultiplicative invariant. At the time, however, little was known about the properties of the rank gradient nor had any computations been carried out. M. Lackenby was the first to study the rank gradient in more detail ([19]), although it was mentioned in [32].

Having positive rank gradient (that is, $RG(G) > 0$) is a strong property. By the Nielsen-Schreier index formula $RG(F_n) = n - 1$. In [19], M. Lackenby proved that if G is a non-trivial free product $A * B$, where either A or B is not isomorphic to C_2 , the cyclic group of order two, then $RG(G) > 0$. On the other hand, $SL_n(\mathbb{Z})$ for $n \geq 3$, ascending HNN-extensions, and direct products of finitely generated infinite residually finite groups all have zero rank gradient ([19], [1]). In some sense, the rank gradient measures how free a group is.

Proposition 2.2 says that having positive rank gradient is invariant under finite index subgroups and finite index supergroups and it shows that it defines a generalised Euler characteristic as mentioned in [38].

The rank gradient is related to the important notion of amenability. This notion has many different definitions. Here, we give the following one. A group G is *amenable* if it has a finitely additive, left invariant, probability measure. Being amenable is invariant under quotients, extensions, subgroups and finite direct products. The notion was first introduced by J. von Neumann. Finite groups, finitely generated abelian groups and soluble groups are examples of amenable groups. On the other hand, any group which contains a non-abelian free group is non-amenable. Regarding the rank gradient, M. Lackenby proved in [19], that if a finitely presented residually finite group is amenable, then it has zero rank gradient. This result was strengthened by M. Abert and N. Nikolov in [1] to finitely generated residually finite groups.

Another important property is Property (τ) . A finitely generated group has *Property (τ)* if for some (equivalently any) finite generating set S for G , the set of Cayley graphs $\text{Cay}(G/N, S)$ form an expander family, where N varies over all finite index normal subgroups of G . This property is preserved under quotients, extensions and subgroups of finite index. It is remarkable that this property has been used in many different areas such as graph theory, differential geometry, group theory and representation theory. Moreover, the theory relating to Property (τ) is very rich (see for instance [22]).

If G is a finitely generated residually finite group which is amenable and has property (τ) , then G is finite ([22]). In [19], M. Lackenby proved that if a group G is finitely presented, then either it has positive rank gradient, it has Property (τ) , or an infinite number of finite index subgroups are HNN extensions or amalgamated free products.

Say G is a finitely presented group given by a finite presentation $P = \langle X | R \rangle$. The deficiency of P is defined as $|X| - |R|$. The deficiency of G is defined as the supremum over all the possible finite presentations P that G admits.

Note. Deficiency is well defined as the deficiency of a finite presentation is bounded above by $\beta_1(G)$, where $\beta_1(G)$, the *first Betti number of G* , is the number of copies of \mathbb{Z} which appear in the abelianisation of G . One way of seeing that the deficiency is bounded above by $\beta_1(G)$ is by computing the abelianisation of the group using Smith's normal form.

Consider a set X of generators for G . Send this set under the canonical surjective homomorphism f from G onto its abelianisation. Then $f(X)$ generates the abelianisation of G . Therefore, $\beta_1(G)$ is a lower bound for $d(G)$. Hence

$$def(G) \leq \beta_1(G) \leq d(G). \quad (11)$$

Consider P a finite presentation for G . Let H be a finite index subgroup of G . The Reidemeister-Schreier rewriting method computes a finite presentation Q for H and shows that

$$def(P) - 1 = \frac{def(Q) - 1}{|G : H|}.$$

As the deficiency of Q is less than or equal to the deficiency of H , then for all finite presentations P of G

$$def(P) - 1 = \frac{def(Q) - 1}{|G : H|} \leq \frac{def(H) - 1}{|G : H|}.$$

As the deficiency of G is the supremum over all finite presentations, then

$$def(G) - 1 \leq \frac{def(H) - 1}{|G : H|}. \quad (12)$$

Therefore, the deficiency minus one of a finitely presented group is a supermultiplicative invariant with respect to finite index subgroups.

Remark 2.3. As mentioned before, having deficiency greater than one is a strong property. As large groups contain a non-abelian free group, then they are non-amenable. Moreover, a large group does not have Property (τ) .

The deficiency of a finitely presented group G and its rank gradient are nicely related. By eq. (11) and eq. (12)

$$def(G) - 1 \leq \frac{def(H) - 1}{|G : H|} \leq \frac{d(H) - 1}{|G : H|},$$

where H is a finite index subgroup of G . Therefore,

$$def(G) - 1 \leq RG(G).$$

Hence, if the finitely presented group G has deficiency greater than one, then the group has positive rank gradient.

Both largeness and having positive rank gradient are properties invariant under finite index supergroups. Therefore, in order to conclude both largeness and positive rank gradient, it is enough to have a finite index subgroup with deficiency greater than one. By eq. (12), we define the *Deficiency Gradient* as

$$DG(G) = \sup_{\substack{H \leq G \\ f}} \left\{ \frac{\text{def}(H) - 1}{|G : H|} \right\}.$$

Note that $DG(G) > 0$ if and only if G has a finite index subgroup with deficiency greater than one. Therefore, having positive deficiency gradient implies the group G is both large and has positive rank gradient.

By Proposition 2.2, this invariant is multiplicative with respect to finite index subgroups and hence defines a generalised Euler characteristic.

In [33] (alternatively [8]), the notion of p -deficiency was introduced. Let G be a finitely generated group given by a presentation $Q = \langle X | R \rangle$, where X generates F_n , the non-abelian group of rank n . The author of [33] defined $\nu_p(r)$ for $r \in R$, to be the largest integer k , such that there is some $s \in F_n$ with $r = s^{p^k}$. The p -deficiency of Q is defined in [8] as

$$\text{def}_p(Q) = n - \sum_{r \in R} p^{-\nu_p(r)}.$$

The definition of p -deficiency in [33] is

$$\text{def}_p(Q) = n - \sum_{r \in R} p^{-\nu_p(r)} - 1.$$

For example, let Q be the following presentation

$$\langle x_1, \dots, x_n \mid w_1^{p^{a_1}}, \dots, w_m^{p^m} \rangle,$$

where w_i cannot be expressed as a proper p -power for all i , $1 \leq i \leq m$. Then, the p -deficiency of Q as defined in [8] is

$$n - \sum_{i=1}^m \frac{1}{p^{a_i}},$$

while the p -deficiency of Q as defined in [33] is

$$n - \sum_{i=1}^m \frac{1}{p^{a_i}} - 1.$$

We opt for the first definition, that is, the one in [8].

We will be interested only in presentations Q such that the quantity $\sum_{r \in R} p^{-\nu_p(r)}$ converges. That is, presentations for which $\text{def}_p(Q)$ is finite.

Given G a finitely generated group with a presentation $Q = \langle X | R \rangle$, where X generates F_n and such that $\text{def}_p(Q)$ is finite, then the p -deficiency of G is defined as the supremum of the p -deficiencies of all presentations of G with finite generating set.

The authors of [4], proved that p -deficiency minus one is supermultiplicative with respect to finite index normal subgroups. Hence,

$$\text{def}_p(G) - 1 \leq \frac{\text{def}_p(H) - 1}{|G : H|},$$

where H is a finite index normal subgroup in G . Therefore, it makes sense to consider the supremum over all normal finite index subgroups. Hence we define the p -deficiency gradient to be

$$DG_p(G) = \sup_{\substack{H \trianglelefteq G \\ \text{f.i.}}} \left\{ \frac{\text{def}_p(H) - 1}{|G : H|} \right\}.$$

By Proposition 2.1 the quantity DG_p would still be the same if the supremum is taken over finite index subgroups.

Having p -deficiency greater than one is also a strong property. For instance, in [8] it is shown that a finitely presented group with p -deficiency greater than one is p -large (a group is p -large if it has a normal subgroup of index a power of p which surjects onto a non-abelian free group).

The deficiency gradient, p -deficiency gradient and rank gradient are nicely related. Consider a finitely presented group G and a finite presentation P for G . Using the definitions of deficiency and p -deficiency for P , the deficiency of P is less than or equal to its p -deficiency. Hence

$$\frac{\text{def}(H) - 1}{|G : H|} \leq \frac{\text{def}_p(H) - 1}{|G : H|},$$

where H is a finite index normal subgroup of G . Therefore, $DG(G) \leq DG_p(G)$.

Both deficiency and p -deficiency are supermultiplicative. Hence the inequality $DG(G) \leq DG_p(G)$ is rather straightforward. The rank gradient is however submultiplicative. Therefore, the following proposition comes as somewhat of a surprise.

Proposition 2.4. *Let G be a finitely generated group. Then $DG_p(G) \leq RG(G)$.*

Proof. In [33], J.C. Schlage-Puchta proved that given a finitely generated group G , then $\text{def}_p(G) \leq d_p(G) \leq d(G)$, where $d_p(G)$, the p -rank of G , is defined as the maximum number of copies of C_p , the cyclic group of order p , onto which G can surject.

Let H be a finite index normal subgroup of G . By the supermultiplicativity of p -deficiency

$$\text{def}_p(G) - 1 \leq \frac{\text{def}_p(H) - 1}{|G : H|} \leq \frac{d_p(H) - 1}{|G : H|} \leq \frac{d(H) - 1}{|G : H|}.$$

This holds for all normal finite index subgroups, therefore $\text{def}_p(G) - 1 \leq RG(G)$ and hence

$$\frac{\text{def}_p(H) - 1}{|G : H|} \leq \frac{RG(H)}{|G : H|} = RG(G),$$

where the last equality follows from Proposition 2.2. Take the supremum on the left hand side, over all finite index normal subgroups of G , to obtain the result. \square

Remark 2.5. From Proposition 2.4, if $\text{def}_p(G) > 1$, then $RG(G) > 0$. This may also be deduced from results in [4]. Also note that having $DG_p(G) > 0$ is equivalent to having a finite index normal subgroup with p -deficiency greater than one. Hence $DG_p(G) > 0$ implies the group G has a finite index normal subgroup H which is p -large and has positive rank gradient. As H has finite index in G , the latter is large and has positive rank gradient.

Corollary 2.6. *Let G be a finitely presented group. Then*

$$DG(G) \leq DG_p(G) \leq RG(G).$$

By Proposition 3.7 in [18], $d_p(G) - 1$ defines a submultiplicative invariant for the class of finitely generated groups, with respect to subnormal subgroups of finite index a power of p . However, it does not define a submultiplicative or supermultiplicative invariant with respect to normal finite index subgroups, even if we restrict ourselves to the class of finitely presented groups.

Consider a prime p and the free product $C_{p_1} * C_{p_2}$, where p_1 and p_2 are primes distinct from p and at least one of p_1 and p_2 is greater than 2. The abelianisation of $C_{p_1} * C_{p_2}$ is $C_{p_1} \times C_{p_2}$, therefore its p -rank is zero. However, $C_{p_1} * C_{p_2}$ has a non-abelian free group F of finite rank as a finite index subgroup.

Any non-abelian free group has positive p -rank for any prime p . Therefore,

$$0 < \frac{d_p(F) - 1}{|C_{p_1} * C_{p_2} : F|}.$$

However, $d_p(C_{p_1} * C_{p_2}) - 1 = -1$. Therefore, the p -rank does not define a submultiplicative invariant with respect to finite index subgroups.

Let p and $G = C_{p_1} * C_{p_2}$ be as above. Consider $\Gamma = G \times C_p$. The p -rank of G is zero. Note G is finite index in Γ . However, the p -rank of Γ is at least one. Therefore $d_p(\Gamma) - 1 \geq 0$, but

$$\frac{d_p(G) - 1}{|\Gamma : G|} < 0.$$

Therefore, the p -rank does not define a supermultiplicative invariant with respect to finite index subgroups.

Even when the p -rank does not define a supermultiplicative or a submultiplicative invariant, with respect to finite index subgroups, the way in which it varies on its finite index subgroups has been useful in some cases. In [20], M. Lackenby considered collections of finite index subgroups $\{G_i\}$ for a finitely presented group G and said that $\{G_i\}$ has linear growth of mod p homology if

$$\inf_i \left\{ \frac{d_p(G_i)}{|G : G_i|} \right\} > 0. \quad (13)$$

M. Lackenby used this condition to give a characterisation for largeness in finitely presented groups ([20]).

In [4], Y. Barnea and J.C. Schlage-Puchta considered the quantities

$$\alpha^-(G) = \liminf \frac{d_p(H)}{|G : H|}$$

and

$$\alpha^+(G) = \limsup \frac{d_p(H)}{|G : H|},$$

where the \liminf and the \limsup are taken over all finite index normal subgroups of G .

The authors of [4] proved that if a finitely generated group G has any of the two quantities above being positive, say $\alpha^-(G)$, then G has a normal subgroup N such that G/N is a p -group and $\alpha^-(G/N) > 0$.

We define the p -rank gradient as

$$RG_p(G) := \inf_{\substack{H \trianglelefteq G \\ f}} \left\{ \frac{d_p(H) - 1}{|G : H|} \right\}.$$

Proposition 2.7. *Let G be a finitely generated group. Then $\text{def}_p(G) - 1 \leq RG_p(G)$. In particular, if $\text{def}_p(G) > 1$, then $RG_p(G) > 0$.*

Proof. In [4] it was proved that p -deficiency minus one is supermultiplicative with respect to finite index normal subgroups of G . That is, if H is a finite index normal subgroup of G , then

$$\text{def}_p(G) - 1 \leq \frac{\text{def}_p(H) - 1}{|G : H|}.$$

Moreover, the p -deficiency of a group is a lower bound for the p -rank of a group ([33]), hence

$$\text{def}_p(G) - 1 \leq \frac{\text{def}_p(H) - 1}{|G : H|} \leq \frac{d_p(H) - 1}{|G : H|}. \quad (14)$$

The result is obtained by taking the infimum over all the finite index normal subgroups of G . \square

As the p -rank gradient and the rank gradient are defined by using the infimum, and $d_p(G) \leq d(G)$, then $RG_p(G) \leq RG(G)$, where G is a finitely generated group. From Proposition 2.7 we then have

$$\text{def}_p(G) - 1 \leq RG_p(G) \leq RG(G).$$

Now we introduce another notion of deficiency which we call group deficiency. This notion is partly inspired by the following result in [36] by R. Thomas.

Theorem 2.8. *(Only Theorem in [36])*

Let G be given by the presentation

$$Q = \langle x_1, \dots, x_n \mid w_1^{m_1}, \dots, w_s^{m_s} \rangle,$$

and let $\varphi : F_n \rightarrow G$ be the canonical map from the non-abelian free group of rank n onto G . If the order of $\varphi(w_i)$ in G is m_i , for all i , $1 \leq i \leq s$, and

$$n - \sum_{i=1}^s \frac{1}{m_i} \geq 1,$$

then the group G is infinite.

This theorem considers the quantity $n - \sum_{i=1}^s 1/m_i$, which is different to the quantities considered in the deficiency and the p -deficiency. This leads to the following definition.

Definition 2.9. Let G be a finitely presented group given by a presentation $Q = \langle x_1, \dots, x_n \mid w_1^{m_1}, \dots, w_s^{m_s} \rangle$. Let $\varphi : F_n \rightarrow G$ be the canonical map from the non-abelian free group of rank n onto G , induced by the presentation Q . Suppose the order of $\varphi(w_i)$ in G is k_i , for all i , $1 \leq i \leq s$. Then we define the *group deficiency* of Q to be

$$n - \sum_{i=1}^s 1/k_i.$$

We define the group deficiency of G as the supremum of all the group deficiencies taken over all possible finite presentations of G .

Note that the group deficiency compares to the deficiency in the following way

$$\text{def}(Q) \leq \text{gdef}(Q).$$

It is not clear whether the group deficiency defines a submultiplicative or supermultiplicative invariant. We can nevertheless define the *Group Deficiency Gradient* as

$$\text{GDG}(G) = \sup_{\substack{H \leq G \\ f}} \left\{ \frac{\text{gdef}(H) - 1}{|G : H|} \right\}.$$

The previous definition implies that

$$\text{DG}(G) \leq \text{GDG}(G). \tag{15}$$

Let G be given by the presentation

$$Q = \langle x_1, \dots, x_n \mid u_i^{m_i} \rangle, \quad i \in I,$$

where I is a countable set. Suppose k_i is the order of the image of u_i in G , for all $i \in I$. If $\sum_{i \in I} 1/k_i$ converges, then it makes sense to define the group deficiency of G as $n - \sum_{i \in I} 1/k_i$. This helps extend the notion beyond the class of finitely presented groups.

Another invariant which is useful to keep track of on finite index subgroups is the first Betti number. In [11], the first L^2 -Betti number $\beta_1^{(2)}(G)$ is defined for a finitely

generated group G . For definition and main properties we refer to [11]. For the moment, it suffices to note that the first L^2 -Betti number satisfies

$$\beta_1^{(2)}(G) = \frac{\beta_1^{(2)}(H)}{|G : H|} \quad (16)$$

for all finite index subgroups H in G . Therefore, it defines a generalised Euler characteristic.

When restricted to finitely presented residually finite groups, Lück's approximation Theorem ([25]) helps us reduce the definition of the first L^2 -Betti number to the following: given a finitely presented residually finite group G and a descending sequence of finite index normal subgroups $\{N_i\}_{i \in \mathbb{N}}$, such that $\bigcap_{i \in \mathbb{N}} N_i = \{e\}$, then

$$\beta_1^{(2)}(G) = \lim_{i \rightarrow \infty} \frac{\beta_1(N_i)}{|G : N_i|}.$$

In [26], the finitely generated case was addressed. It turns out from [26] that if the group G is finitely generated and residually finite, then

$$\beta_1^{(2)}(G) \geq \limsup_{i \rightarrow \infty} \frac{\beta_1(N_i)}{|G : N_i|}. \quad (17)$$

Moreover, the inequality cannot be improved as examples where the strict inequality holds are given in [26].

The authors of [17] used the group deficiency of a finitely generated group to give the following result

Theorem 2.10. ([17], Theorem 4)

Suppose a group G is given by the presentation

$$Q = \langle x_1, \dots, x_n \mid u_i^{m_i} \rangle, \quad i \in I,$$

where I is a countable set and the order of u_i in G is m_i for all $i \in I$. Suppose $\sum_{i \in I} 1/m_i$ converges. If G is infinite, then

$$\beta_1^2(G) \geq n - 1 - \sum_{i \in I} \frac{1}{m_i}.$$

Corollary 2.11. Let G be a finitely presented group G . Then

$$GDG(G) \leq \beta_1^2(G).$$

Proof. Let H be a finite index subgroup of G . As G is finitely presented, then H is finitely presented. Therefore, by the previous theorem we have

$$\beta_1^2(H) \geq gdef(H) - 1.$$

Since the first L_2 -Betti number is multiplicative with respect to finite index subgroups

$$\beta_1^2(G) = \frac{\beta_1^2(H)}{|G : H|} \geq \frac{gdef(H) - 1}{|G : H|}.$$

The result is obtained by taking the supremum, on the right hand side, over all finite index subgroups of G . □

Corollary 2.12. *Let G be a finitely presented group. Then*

$$DG(G) \leq GDG(G) \leq \beta_1^2(G).$$

Proof. Immediate from eq. (15) and Corollary 2.11. □

As mentioned before, the rank gradient, the deficiency gradient and the first L^2 -Betti number define generalised Euler characteristics. The authors of [4] considered

$$\chi_p(G) = - \sup_{|G:H| < \infty} \frac{\text{def}_p(H) - 1}{|G : H|}$$

and proved the following result.

Proposition 2.13. ([4], Proposition 17).

Let G be a finitely generated group.

1. *Then*

$$\chi_p(G) = - \sup_{\substack{|G:H| < \infty \\ H \triangleleft G}} \frac{\text{def}_p(H) - 1}{|G : H|}.$$

2. *If H is a finite index subgroup of G , then $\chi_p(H) = |G : H| \chi_p(G)$.*

3. *If G is virtually free, then $\chi_p(G)$ is the ordinary Euler characteristic of G .*

4. *If G is a Fuchsian group, then $\chi_p(G)$ is the ordinary Euler characteristic of G .*

The following propositions show that the deficiency gradient, rank gradient and first L^2 -Betti number, satisfy the same properties as $\chi_p(G)$. Note that the first two properties have already been settled in Proposition 2.1 and Proposition 2.2, hence only the last two are proved.

Proposition 2.14. *Let G be a finitely presented group. Then if G is virtually free or a Fuchsian group, then $-DG(G)$ is the ordinary Euler characteristic of G .*

Proof. Denote the ordinary Euler characteristic of G by $\chi(G)$. The Euler characteristic and the deficiency gradient are multiplicative. Therefore, when G is virtually free, it suffices to prove the statement for non-abelian free groups. Also, as every Fuchsian group has a surface group as a finite index subgroup, it suffices to prove the statement for surface groups when G is a Fuchsian group.

Suppose G is free of rank m . Then the deficiency of G is m . Let H be a finite index subgroup of G . By the Nielsen-Schreier index formula H is a free group of rank $|G : H|(m - 1) + 1$ and hence has deficiency $|G : H|(m - 1) + 1$. Therefore,

$$-DG(G) = - \sup_{\substack{H \leqslant G \\ f}} \left\{ \frac{\text{def}(H) - 1}{|G : H|} \right\} = - \sup_{\substack{H \leqslant G \\ f}} \left\{ \frac{1 + |G : H|(m - 1) - 1}{|G : H|} \right\}$$

$$= 1 - m = \chi(G).$$

Let G be $\pi_1(S_g)$, where S_g is the orientable surface of genus $g > 1$. The deficiency of $\pi_1(S_g)$ is $2g - 1$. If H is a finite index subgroup of G , then $H \cong \pi_1(S_h)$, where $h = |G : H|(g - 1) + 1$. For $l > 1$, the Euler characteristic of S_l is $(2 - 2l)$. Therefore,

$$\begin{aligned} -DG(G) &= -\sup_{\substack{H \leq G \\ f}} \left\{ \frac{\text{def}(H) - 1}{|G : H|} \right\} \\ &= -\sup_{\substack{H \leq G \\ f}} \left\{ \frac{\left(2(|G : H|(g - 1) + 1) - 1 \right) - 1}{|G : H|} \right\} = 2 - 2g. \end{aligned}$$

□

Proposition 2.15. *Let G be a finitely generated group. If G is virtually free or a Fuchsian group, then $-RG(G)$ is the ordinary Euler characteristic of G .*

Proof. If G is virtually free, then the proof in Proposition 2.14 works equally well. Just note that if G is free of rank m then the deficiency of G is m .

Let G be a Fuchsian group. Suppose, as in Proposition 2.14, that $G \cong \pi_1(S_g)$, $g > 1$. The rank of $\pi_1(S_g)$ is $2g$. If H is a finite index subgroup of G , then $\pi_1(S_h) \cong H$, where $h = |G : H|(g - 1) + 1$. Therefore, the rank of H is $2h$. Hence,

$$\begin{aligned} -RG(G) &= -\inf_{\substack{H \leq G \\ f}} \left\{ \frac{d(H) - 1}{|G : H|} \right\} = -\inf_{\substack{H \leq G \\ f}} \left\{ \frac{2(|G : H|(g - 1) + 1) - 1}{|G : H|} \right\} \\ &= -\inf_{\substack{H \leq G \\ f}} \left\{ \frac{2 + 2|G : H|(g - 1) - 1}{|G : H|} \right\} = 2 - 2g. \end{aligned}$$

□

Proposition 2.16. *Let G be a finitely generated group. If G is virtually free or a Fuchsian group, then $-\beta_1^{(2)}(G)$ is the ordinary Euler characteristic of G .*

Proof. Once again, it suffices to show the statement for non-abelian free groups and surface groups. Let F_k be the non-abelian free group of rank k . Consider a descending sequence of finite index normal subgroups in F_k that intersects in the identity. This can be done as F_k is residually finite. Note that N_i is a non-abelian free group, therefore, $\beta_1(N_i) = d(N_i) = |F_k : N_i|(k - 1) + 1$. Hence,

$$-\beta_1^{(2)}(F_k) = -\lim_{i \rightarrow \infty} \frac{\beta_1(N_i)}{|F_k : N_i|} = -\lim_{i \rightarrow \infty} \frac{|F_k : N_i|(k - 1) + 1}{|F_k : N_i|} = 1 - k = \chi(F_k).$$

Suppose $G \cong \pi_1(S_g)$ with $g > 1$. Consider a descending sequence $\{G_i\}_{i \in \mathbb{N}}$ of finite index normal subgroups in G that intersects in the identity. Such series exist as G is residually finite. Note that since G is a surface group, then G_i is a surface group for all $i \in \mathbb{N}$.

The Euler characteristic is multiplicative with respect to finite index subgroups. Therefore, $2g_i - 2 = |G : G_i|(2g - 2)$ where $g_i = |G : G_i|(g - 1) + 1$. Finally, note that $\beta_1(G_i) = 2g_i$. Therefore,

$$-\lim_{i \rightarrow \infty} \frac{\beta_1(G_i)}{|G : G_i|} = -\lim_{i \rightarrow \infty} \frac{2g_i}{|G : G_i|} = -\lim_{i \rightarrow \infty} \frac{2|G : G_i|(g - 1) + 2}{|G : G_i|} = 2 - 2g.$$

□

3 Residual deficiency

This section introduces an invariant for finitely presented groups that we call *residual deficiency*. From now on, unless otherwise stated, any group considered is finitely presented. If G is a finitely presented group and $Q = \langle X | R \rangle$ is a finite presentation for G , where X freely generates F_n , the non-abelian free group of rank n , denote by φ the canonical homomorphism from F_n to G . Denote by R_G the finite residual of G , that is, the intersection of all finite index subgroups of G , and call G/R_G the *residual quotient* of G .

Definition 3.1. Let G be a finitely presented group and let $Q = \langle X | R \rangle$ be a finite presentation for G . Consider the elements of R expressed in the following way $R = \{r_1 = u_1^{m_1}, \dots, r_s = u_s^{m_s}\}$, where the elements $u_i \in F_n$ cannot be expressed as proper powers of other words in F_n . Let \overline{R}_G be the inverse image of R_G under φ and let k_i be the order of u_i in $F_n/\overline{R}_G (\cong G/R_G)$. We define the *residual deficiency of the finite presentation Q* to be

$$rdef(Q) = n - \sum_{i=1}^s \frac{1}{k_i}.$$

Note that k_i is also equal to the order of $\varphi(u_i)$ in G/R_G .

Consider all possible finite presentations for G and their corresponding residual deficiencies. We define the *residual deficiency* of G as the supremum of these residual deficiencies,

$$rdef(G) = \sup_{\langle X | R \rangle \cong G} \{rdef(Q)\}.$$

Remark 3.2. The first thing to observe is that if Q is a finite presentation for a group G , then $def(Q) \leq rdef(Q)$. Hence, $def(G) \leq rdef(G)$. However, if the residual quotient of G is torsion free, then $def(G) = rdef(G)$. This is because the only way in which the residual deficiency may be greater than the deficiency is if there is a presentation in which one of the relators is expressed as a proper power of another element which lies outside of \overline{R}_G . This element then defines a non-trivial torsion element in the residual quotient.

The following lemma (or variations of it) is a well known result that appears in many papers, for instance Lemma 2.3 of [30].

Lemma 3.3. *Let G be a finitely generated group, H a normal subgroup of finite index in G , and g an element of H . Denote the centraliser of g in G by $C_G(g)$. Consider K*

a subgroup of $C_G(g)$. Let T be a left transversal for HK in G and let Z be the set of conjugates of g by elements in T . Then, the normal subgroup in G generated by g , is the normal subgroup in H generated by Z .

Proof. Suppose HK/H has order n . Note that $H \trianglelefteq HK \leq G$. Therefore, if H has index j in G , then HK has index j/n in G .

The proof centres around showing that if $T = \{t_1, \dots, t_{j/n}\}$ is a left transversal for HK in G , then $\langle\langle g \rangle\rangle^G = \langle\langle t_l g t_l^{-1} \rangle\rangle^H$, where $1 \leq l \leq j/n$. For this it suffices to show that $s g s^{-1}$ is in $\langle\langle t_l g t_l^{-1} \rangle\rangle^H$, for all s in G .

Note that s can be written as $t_l h k$ for some t_l in T , h in H and k in K , where K is in the centraliser of g in G . Hence,

$$s g s^{-1} = t_l h k g k^{-1} h^{-1} t_l^{-1} = t_l h g h^{-1} t_l^{-1} = h' t_l g t_l^{-1} h'^{-1}$$

where $h' = t_l h t_l^{-1}$ as H is normal in G . Thus $s g s^{-1} \in \langle\langle t_l g t_l^{-1} \rangle\rangle^H$. \square

The following Theorem is central in this paper. Similar forms of this result may be found in the literature. For instance in [12] and in [6].

Theorem 3.4. *Let the group G be given by the presentation*

$$P = \langle x_1, \dots, x_n \mid u_1^{m_1}, \dots, u_r^{m_r} \rangle,$$

where $n, m_i \geq 1$ for $i = 1, \dots, r$. Let k_i , for $1 \leq i \leq r$, be as in Definition 3.1, the order of $\varphi(u_i)$ in G/R_G . Then, there are finite index normal subgroups H in G , such that the order of $\varphi(u_i)$ in G/H is k_i , for all i , $1 \leq i \leq r$. Moreover, the deficiency of every such H is bounded below by

$$1 + |G : H|(r \operatorname{def}(P) - 1).$$

Proof. As P is a presentation for G , there is a surjective homomorphism φ from F_n , the non-abelian free group of rank n , onto G , where the kernel N is the normal subgroup of F_n generated by $u_1^{m_1}, \dots, u_r^{m_r}$.

Denote by \overline{R}_G the inverse image of R_G under φ . As $F_n/\overline{R}_G \cong G/R_G$, then F_n/\overline{R}_G is residually finite. Therefore, for each word $u_i^{m_i}$, there is a finite index normal subgroup \overline{H}_i in F_n , such that $o(u_i, F_n/\overline{R}_G) = o(u_i, F_n/\overline{H}_i)$. Let \overline{H} be the intersection of all the \overline{H}_i . Then, as there are only finite number of relators in P , the subgroup \overline{H} has finite index in F_n/\overline{R}_G and $o(u_i, F_n/\overline{R}_G) = o(u_i, F_n/\overline{H})$ for all i , $1 \leq i \leq r$. Note that every finite index normal subgroup in G contained in H also satisfies this condition.

Denote by H the normal subgroup in G that corresponds to \overline{H} under φ . Note that $|G : H| = |F_n : \overline{H}|$. Compute a presentation for H using the Reidemeister-Schreier rewriting process (pp. 103, [27]). This gives a presentation with $|G : H|(n - 1) + 1$ generators which freely generate \overline{H} . To compute the relators note that the subgroup N is generated as a normal subgroup of \overline{H} by elements of the type $s u_i^{m_i} s^{-1}$, where $s \in T$ and T is a transversal of \overline{H} in F_n . Since $\overline{H} \leq \overline{H}\langle u_i \rangle \leq F_n$, then every s in T can be written as $t h u_i^l$, with $t \in T_i$, where T_i a left transversal of $\overline{H}\langle u_i \rangle$ in F_n , $h \in \overline{H}$, and $0 \leq l < o(u_i, F_n/\overline{H})$. Moreover, the subgroup in F_n generated by u_i is a subgroup

of $C_{F_n}(u_i)$. Therefore, by Lemma 3.3, each element $u_i^{m_i}$ only needs to be conjugated by $|G : H|/o(u_i, F_n/\overline{H})$ elements. Hence, H admits a presentation with deficiency

$$|G : H|(n - 1) + 1 - \sum_{i=1}^r \frac{|G : H|}{o(u_i, F_n/\overline{H})} = 1 + |G : H|(rdef(P) - 1).$$

□

Corollary 3.5. *Let G be a finitely presented group. Then $rdef(G) - 1 \leq DG(G)$. In particular, if $rdef(G) > 1$, then G has a finite index subgroup with deficiency greater than one.*

Proof. By Theorem 3.4

$$rdef(P) - 1 \leq \frac{def(H) - 1}{|G : H|} \leq DG(G).$$

Taking the supremum over all finite presentations P of G yields

$$rdef(G) - 1 \leq DG(G).$$

If $rdef(G) > 1$, then $DG(G) > 0$, which is equivalent to having a finite index subgroup with deficiency greater than one. □

Remark 3.6. From eq. (15) $DG(G) \leq GDG(G)$, therefore,

$$rdef(G) - 1 \leq DG(G) \leq GDG(G).$$

Example 3.7. This example shows that the equality $rdef(G) - 1 \leq DG(G)$ is not a strict equality in general. Consider the Thompson group T given by the following presentation

$$\langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle. \quad (18)$$

One of the main properties of this group is that its commutator subgroup is equal to its residual subgroup. Such groups are known as residually abelian. For this and other properties of the Thompson group we refer to [9].

As the relations in eq. (18) are in the commutator, the abelianisation of T is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Since the commutator subgroup of T is the same as its residual subgroup, then the commutator $[a, b]$ generates \overline{R}_T as a normal subgroup of F_2 .

Consider $G := T * \mathbb{Z}$. Denote the normal closure of $[a, b]$ in $F_3 = \langle a, b, c \rangle$ by N . Since \overline{R}_G is normal in F_3 and $[a, b] \in \overline{R}_T \trianglelefteq \overline{R}_G$, then $\overline{R}_T \trianglelefteq N \trianglelefteq \overline{R}_G$. A presentation for G/N is given by

$$\langle a, b, c \mid [a, b] \rangle.$$

This defines a group isomorphic to $(\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$. Since $N \trianglelefteq \overline{R}_G$ and $G/N \cong (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$ is residually finite, then $N = \overline{R}_G$. As G/\overline{R}_G is isomorphic to $(\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$, which is torsion free, then $def(G) = rdef(G)$ by Remark 3.2. In [7], J. Button showed that $def(G) = 1$. Therefore, $rdef(G) = 1$.

Finally, the Kurosch subgroup theorem allows us to conclude that $T * \mathbb{Z} * \mathbb{Z}$ is a subgroup of finite index in G . Since $T * \mathbb{Z} * \mathbb{Z}$ has deficiency greater than one, then G has a finite index subgroup with deficiency greater than one and hence $DG(G) > 0$. Therefore, G satisfies

$$rdef(G) - 1 < DG(G).$$

Theorem 3.4 implies the following result by D. Allcock ([3]).

Corollary 3.8. ([3], *only theorem*).

Let the group G be given by the presentation

$$P = \langle x_1, \dots, x_n \mid u_1^{m_1}, \dots, u_r^{m_r} \rangle.$$

Suppose H is a normal subgroup of G of index $N < \infty$, and that for each j , $1 \leq j \leq r$, we have $u_j^k \notin H$ for $k = 1, \dots, m_j - 1$. Then the rank of the abelianisation of H is at least

$$1 + |G : H|(n - 1 - \sum_{j=1}^r \frac{1}{m_j}).$$

Proof. By Theorem 3.4, the deficiency of H is at least

$$1 + |G : H|(rdef(P) - 1),$$

where the latter is greater than or equal to

$$1 + |G : H|(n - 1 - \sum_{j=1}^r \frac{1}{m_j}).$$

The result follows as the deficiency is a lower bound for the rank of the abelianisation. \square

Remark 3.9. In [3], Corollary 3.8 is used to conclude Theorem 2.8 proved by R. Thomas in [36]. As Theorem 3.4 implies Corollary 3.8, then Theorem 3.4 also implies Theorem 2.8.

Corollary 3.10. *Let G be a finitely presented group. Then*

$$rdef(G) - 1 \leq DG(G) \leq DG_p(G) \leq RG(G),$$

$$rdef(G) - 1 \leq DG(G) \leq GDG(G) \leq \beta_1^2(G).$$

In particular, if the residual deficiency of G is greater than one, then G has positive rank gradient, positive first L^2 -Betti number and is large.

Proof. The inequalities follow from Corollary 2.6, Corollary 2.12 and Corollary 3.5. Positive rank gradient and first L^2 -Betti number are immediate from the inequalities. For largeness, if $rdef(G) > 1$, then $DG(G) > 0$, which is equivalent to having a finite index subgroup with deficiency greater than one. The latter implies largeness by [5]. \square

The residual deficiency of a finite presentation can easily be computed or at least approximated in some cases. To illustrate this, we first need the following definition and lemma.

Definition 3.11. We say that the finite presentation

$$P = \langle x_1, \dots, x_n \mid r_1^{a_1}, \dots, r_m^{a_m} \rangle$$

has a *quotient with no collapse* if G , the group defined by P , has a homomorphism ψ from G to a group N , such that the order of $\psi \circ \varphi(r_i)$ in N is a_i , for all i , $1 \leq i \leq m$, where φ is the canonical map induced by P from F_n to G .

Consider the particular case when N is residually finite. In this case we say G has a *residually finite quotient with no collapse*.

Lemma 3.12. *Let G be given by a presentation $P = \langle x_1, \dots, x_n \mid r_1^{a_1}, \dots, r_m^{a_m} \rangle$, where the words r_i are not expressed as proper powers, for $1 \leq i \leq m$. Suppose G has a residually finite quotient with no collapse ψ . Then*

$$rdef(P) = n - \sum_{i=1}^m \frac{1}{a_i}.$$

Proof. Let K be the kernel of ψ . The condition that the order of $\psi \circ \varphi(r_i)$ in N is a_i says that the order of the image of $\varphi(r_i)$ in G/K is a_i . As the image of ψ is residually finite, the quotient G/K is residually finite and hence the finite residual R_G is contained in K . Therefore, the order of the image of $\varphi(r_i)$ in G/R_G is a_i for all i , $1 \leq i \leq m$. Since the words r_i cannot be expressed as proper powers, then the residual deficiency of P is $n - \sum_{i=1}^m 1/a_i$. \square

Example 3.13. Consider generalised triangle groups. These are groups given by presentations of the form

$$\langle a, b \mid a^l, b^m, w^n \rangle,$$

where

$$w = a^{r_1} b^{s_1} \dots a^{r_k} b^{s_k} \quad (k \geq 1, 0 < r_i < l, 0 < s_i < m, \text{ where } 1 \leq i \leq k).$$

Let G be a generalised triangle group. In [6], a representation ρ from G to $PSL(2, \mathbb{C})$ was constructed, such that the orders of a , b and w in $PSL(2, \mathbb{C})$ are l , m and n , respectively. The image of G in $PSL(2, \mathbb{C})$ is finitely generated. By Mal'cev ([28]), every finitely generated subgroup of $PSL(2, \mathbb{C})$ is residually finite. Hence, the hypotheses of Lemma 3.12 are satisfied. Therefore, if $P = \langle a, b \mid a^l, b^m, w^n \rangle$, its residual deficiency is $2 - 1/l - 1/m - 1/n$, which is greater than one whenever $1/l + 1/m + 1/n < 1$.

In [6] these groups were shown to have a finite index subgroup with deficiency greater than one whenever $1/l + 1/m + 1/n < 1$. The arguments used are very similar to the arguments in Lemma 3.3 and Theorem 3.4. However, for part of the argument, the authors of [6] rely on the fact that by removing the relator w^n one ends up with a free product of finite cyclic groups, whereas Lemma 3.3 and Theorem 3.4 do not.

Example 3.14. A generalisation of the technique used in [6] was used in [14] to prove that one-relator quotients of free products of non-trivial finite cyclic groups satisfy the Freiheitssatz. The type of groups dealt with in [14] are, more specifically, groups given by

$$P = \langle x_1, \dots, x_n \mid x_1^{m_1}, \dots, x_n^{m_n}, w^s \rangle, \quad (19)$$

where $m_i, s > 1$, for all i , $1 \leq i \leq n$, and w is a word that involves all the generating elements x_1, \dots, x_n . The Freiheitssatz condition in this case means that the subgroup generated by any $n - 1$ elements from the generating set is the free product of the cyclic groups corresponding to the $n - 1$ chosen elements. For instance, if the set of $n - 1$ elements from the generating set consists of x_1, \dots, x_{n-1} , then they generate a subgroup in G isomorphic to $C_{m_1} * \dots * C_{m_{n-1}}$.

This result was obtained by showing that G admits a representation into $PSL(2, \mathbb{C})$ where the image of x_i and the image of w in $PSL(2, \mathbb{C})$ have order m_i and s , respectively, for all i , $1 \leq i \leq n$. Although not pointed out in [14], this result is useful to prove that a substantial collection of these groups have a finite index subgroup with deficiency greater than one and hence are large. For this, it suffices to apply Lemma 3.12. The residual deficiency of P is $n - \sum_{i=1}^n 1/m_i - 1/s$ and by Theorem 3.4, whenever this quantity is greater than one, the group defined by P will have a finite index subgroup with deficiency greater than one. The residual deficiency of P is greater than one, for instance, for all $n \geq 4$, since $m_i, s > 1$, for all i , $1 \leq i \leq n$. If $n = 3$, then one of m_1, m_2, m_3 or s must be greater than two.

Example 3.15. A Coxeter group is a group G given by a presentation of the type

$$\langle a_1, \dots, a_n \mid a_1^2, \dots, a_n^2, (a_i a_j)^{m_{ij}}, 1 \leq i < j \leq n \rangle,$$

where $m_{ij} \geq 2$. The value m_{ij} may be ∞ in which case the word $(a_i a_j)^{m_{ij}}$ is omitted from the set of relators.

Since Coxeter groups are finitely generated and have faithful representations into $GL(n, \mathbb{R})$ (Appendix D in [10]), they are residually finite ([28]). Moreover, due to the solution of the word problem in Coxeter groups, the words $a_1, \dots, a_n, a_i a_j$, define non-trivial elements of order 2 and m_{ij} in the group, respectively ([10], chapter 3, section 4). Hence, every Coxeter group G can be surjected onto itself to obtain a residually finite quotient with no collapse. Applying Lemma 3.12 gives that the residual deficiency of the presentation above for a Coxeter group is $n/2 - \sum_{i < j} m_{ij}^{-1}$.

It is known that Coxeter groups are either virtually abelian or large ([10], Theorem 14.1.2). In fact, there is a way of telling, given a Coxeter presentation, whether the group is large or not. For this, consider the *Coxeter diagram* associated to the Coxeter group. Say the given Coxeter group is

$$\langle a_1, \dots, a_n \mid a_1^2, \dots, a_n^2, (a_i a_j)^{m_{ij}}, 1 \leq i < j \leq n \rangle.$$

The Coxeter diagram associated to this presentation is a graph with n vertices where two vertices, say (i, j) , are joined by an edge if and only if $m_{ij} \geq 3$. Consider the connected components of the graph. These are called the irreducible components of the Coxeter diagram. These components are Coxeter diagrams of other Coxeter groups. If

all these components correspond to either spherical Coxeter groups (i.e. finite Coxeter groups), or Euclidean Coxeter groups, both of which have been completely classified ([10], Appendix C), then the group is virtually abelian. However, if one of the components of the Coxeter diagram is not either spherical or Euclidean, then the group is large.

We already know that having a finite index subgroup with deficiency greater than one is a stronger property than largeness. Hence, this property may be used to study large Coxeter groups. There are plenty of examples of Coxeter groups with a finite index subgroup that has deficiency greater than one; given n , it suffices to have $n/2 - \sum_{i < j} m_{ij}^{-1} > 1$. If, for instance, $m_{ij} > n(n-1)/(n-2)$, for all m_{ij} which appear as powers in the presentation, then a quick computation shows that the residual deficiency is greater than one. On the other hand, the class of Coxeter groups which are fundamental groups of 3-dimensional hyperbolic orbifolds, although large (by [2]), cannot have finite index subgroups with deficiency greater than one ([13]).

Example 3.16. Consider the presentation

$$\langle x_1, x_2, x_3 \mid x_1^{e_1}, x_2^{e_2}, x_3^{e_3}, R_1^m, R_2^p, R_3^q \rangle, \quad (20)$$

where $e_i = 0$ or $e_i \geq 2$ for $i = 1, 2, 3$; $2 \leq m, p, q$; R_1 is a cyclically reduced word in the free product on x_1, x_2 which involves both x_1 and x_2 but not x_3 , R_2 is a cyclically reduced word in the free product on x_1, x_3 which involves both x_1 and x_3 but not x_2 and R_3 is a cyclically reduced word in the free product on x_2, x_3 which involves both x_2 and x_3 but not x_1 . Further, each R_i , $i = 1, 2, 3$ is not a proper power in the free product on the generators it involves.

According to the authors of [15], E. Vinberg called a group *generalised tetrahedral* if it is defined by a presentation such as eq. (20).

Let G be a generalised tetrahedral group. In [15], a representation $\rho : G \rightarrow PSL(2, \mathbb{C})$ is constructed where $\rho(x_i)$ has infinite order if $e_i = 0$ or exact order e_i if $e_i \geq 2$ for $i = 1, 2, 3$, and $\rho(R_1)$ has order m , $\rho(R_2)$ has order p , and $\rho(R_3)$ has order q . The authors of [15] call such a representation *essential*. In this case, an essential representation is the same as a quotient with no collapse. Therefore, by Lemma 3.12, the residual deficiency of G is greater than one if

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} + \frac{1}{m} + \frac{1}{p} + \frac{1}{q} < 2, \quad (21)$$

where $1/e_i$ is 0 in case $e_i = 0$.

The authors of [15] use the representation ρ to prove that if eq. (21) holds then G has a finite index subgroup with deficiency greater than one. They do so by using Lemma 3 in [6], which is a particular case of Lemma 3.3. Lemma 3 in [6] uses the fact that when we remove R_1^m, R_2^p and R_3^q from eq. (20), we end up with a free product of finite cyclic groups. Theorem 3.4 does not use this fact.

The authors of [15] mention that their methods also give essential representations for groups defined by the following presentations:

$$\langle x_1, \dots, x_n \mid x_1^{e_1}, \dots, x_n^{e_n}, R_1^{m_1}(x_1, x_2), R_2^{m_2}(x_2, x_3), \dots, R_n^{m_n}(x_n, x_1) \rangle, \quad (22)$$

where $e_i = 0$ or $e_i \geq 2$ for $i = 1, \dots, n$; $m_i \geq 3$ for $i = 1, \dots, n$, and for $i = 1, \dots, n$, $R_i(x_i, x_{i+1})$ is a cyclically reduced word in the free product on x_i, x_{i+1} which involves both x_i and x_{i+1} . And

$$\langle x_1, \dots, x_n \mid x_1^{e_1}, \dots, x_n^{e_n}, S_1^{m_1}(x_1, x_2), \dots, S_{n-1}^{m_{n-1}}(x_1, x_n), S_n^{m_n}(x_2, x_3), \dots, S_{2n-3}^{m_{2n-3}}(x_2, x_n) \rangle, \quad (23)$$

where $n \geq 3$, $e_i = 0$ or $e_i \geq 2$ for $i = 1, \dots, n$, $m_j \geq 3$ for $j = 1, \dots, 2n-3$, and for $j = 1, \dots, n-1$, S_j is a cyclically reduced word in the free product on x_1, x_j involving both x_1 and x_j . Moreover, for $j = 0, \dots, n-3$, S_{n+j} is a cyclically reduced word in the free product on x_2 and x_{j+3} involving both x_2 and x_{j+3} .

The authors of [15] do not mention whether the groups defined by eq. (22) and eq. (23) have a finite index subgroup with deficiency greater than one. However, the residual deficiency of these presentations is easily computed and hence by Theorem 3.4, if the residual deficiency of eq. (22) or eq. (23) is greater than one, then the group it defines has a finite index subgroup with deficiency greater than one and hence are large.

Some of the presentations above are relevant to the results in [12]. In [12], M. Edjvet gave conditions under which balanced presentations define large groups. The balanced presentations considered needed to have at least two proper power relators. The proof was divided into two cases. The first case deals with balanced presentations which define groups with finite abelianisation. The second, when the presentations define groups with infinite abelianisation.

If the group has finite abelianisation and at least one of the powers from the relators is greater than two, then the commutator subgroup has deficiency greater than one. If the group has infinite abelianisation and two of the powers in the relators are not coprime, then the group is large. This case uses results in [35] which only ensure largeness and not the existence of a finite index subgroup with deficiency greater than one.

Consider the family of presentations given by eq. (20) and eq. (22). Suppose $e_1 = e_2 = e_3 = 0$ for the presentations in eq. (20) and $e_i = 0$, $i = 1, \dots, n$, for the presentations in eq. (22). By the results in [15] the presentations given by eq. (20) have residual deficiency greater than or equal to $3/2$, which is greater than one. Since $m_i \geq 3$, $1 \leq i \leq n$, for the presentations given by eq. (22), the residual deficiency is greater than or equal to $n - n/3$, which is greater than one if $n > 1$. Therefore, a presentation in either of these two families which defines a group with infinite abelianisation, is an example of a group given by a balanced presentation which has a finite index subgroup with deficiency greater than one. Moreover, one does not need powers from the relators to have a common prime factor in order to establish largeness.

It is not difficult to see that the families described above have many presentations that define groups with infinite abelianisation. It just suffices to have a single row or column of zeroes in its abelianisation matrix. We define the (i, j) entry of the abelianisation matrix of a presentation $P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ as the power sum of x_j in r_i , which we denote by $\sigma_{x_j}(r_i)$. For example, for presentations given by eq. (20), if $\sigma_{x_1}(R_1) = \sigma_{x_1}(R_2) = 0$, then we get zeroes in the first column. For

presentations given by eq. (22), the abelianisation matrix only has non-zero elements on the diagonal or above it, with the exception of the $(n, 1)$ entry corresponding to $\sigma_{x_1}(R_n^{m_n})$. If $\sigma_{x_1}(R_n) = 0$ and $\sigma_{x_i}(R_i) = 0$ for some i , where $i = 1, \dots, n$, then the abelianisation matrix reduces down to a upper triangular matrix with at least one zero in its diagonal. The determinant of such matrix is therefore 0. Since the absolute value of the determinant of such matrix gives the index of the commutator subgroup in the group (index 0 corresponding to infinite index), then the group defined by such presentation has infinite abelianisation.

Example 3.17. Consider m non-trivial elements u_1, \dots, u_m in F_n , the non-abelian free group of rank n . Denote by $\{a_1, \dots, a_{n-1}, t\}$, the canonical free generators of F_n . Suppose $\sigma_t(u_i) = s_i \neq 0$, for $1 \leq i \leq m$. Define $k = |s_1 \cdots s_m|$ and $n_i = |s_1 \cdots s_{i-1} s_{i+1} \cdots s_m|$. Then the group G given by the presentation

$$P = \langle a_1, \dots, a_{n-1}, t \mid u_1^{n_1}, \dots, u_m^{n_m} \rangle,$$

surjects onto C_k , the cyclic group of order k , by sending a_i to 0, for all i , $1 \leq i \leq n-1$, and t to 1. The residual deficiency of G is bounded as follows

$$rdef(G) \geq rdef(P) \geq n - \sum_{i=1}^m \frac{1}{n_i}.$$

The above expression is greater than one if the values $|s_i|$ are sufficiently large. This construction gives examples of finitely presented groups of arbitrarily large negative deficiency which have finite index subgroups with deficiency greater than one.

Example 3.18. Consider the following deficiency one presentation where $n \geq 2$

$$P = \langle a_1, \dots, a_n \mid u_1^{s_1}, \dots, u_{n-1}^{s_{n-1}} \rangle,$$

such that there is a prime p that divides all the powers s_i , where u_i is a non-trivial word in F_n , for all i , $1 \leq i \leq n-1$. As F_n is residually finite p , then there is a normal subgroup H in F_n , with index in F_n a power of p , such that $u_{i_0} \notin H$, for some i_0 , $1 \leq i_0 \leq n-1$, and has minimal index with respect to this property. This means that if K is a finite index normal subgroup of F_n with index a power of p strictly less than $|F_n : H|$, then $u_i \in K$ for all i , $1 \leq i \leq n-1$.

As the quotient F_n/H is a p -group, it has non-trivial centre. Take a subgroup M of order p in the centre. Denote by L the pullback of M in F_n under the canonical surjective homomorphism from F_n to F_n/H . Note that L is normal in F_n . As the index of L in F_n is less than the index of H in F_n , then $u_i \in L$, $1 \leq i \leq n-1$. This means that u_{i_0} and any other u_i not in H , has order p in F_n/H . Hence the residual deficiency of P is bounded as follows

$$rdef(P) \geq n - (n-2) - \frac{1}{p} > 1.$$

Therefore, the group defined by P has a finite index subgroup with deficiency greater than one. Note that since p divides all powers s_i , then the group has p -deficiency

greater than one and hence it is large ([8]). However, this result does not tell us that the group has a finite index subgroup with deficiency greater than one.

It is known by [35], that all deficiency one presentations with a proper power relator define large groups. However, the question of whether such presentations define groups which have finite index subgroups with deficiency greater than one is still open. The previous example shows that if the deficiency one presentation only has proper power relators and the powers are all divided by a common prime, then the group defined has a finite index subgroup with deficiency greater than one. In particular, if the presentation has two generators and one relator which is a proper power, then the group it defines has a finite index subgroup with deficiency greater than one.

4 Relative size of subgroups and the supermultiplicativity of residual deficiency

The definitions and results in this section follow similar lines to the ones in Section 2 of [4].

Definition 4.1. Let G be a group, K a normal subgroup of G and g an element in G . Define the *relative size of g in G with respect to K* as follows.

First suppose $g \in K$. Define $\nu(g; G, K)$ to be the supremum over $\{o(\psi(a), G/K)\}$, where $a \in G$ is such that $a^n = g$ for some integer n , and where $\psi : G \rightarrow G/K$.

If $g \notin K$, consider all possible expressions of g as a^n where n is a non-zero integer. In this case, define $\nu(g; G, K)$ as the supremum over all integers arising in such expressions.

The relative size of g in G with respect to K is defined as $\nu(g; G, K)^{-1}$.

Note. We say that the value $\nu(g; G, K)$ is infinite if the set $\{o(\psi(a), G/K)\}$ is not bounded. That is, if for every $\varepsilon > 0$, there is $a \in G$ such that $a^n = g$, for some integer n , and $o(\psi(a), G/K) \geq \varepsilon$. In this case, the relative size of g in G with respect to K is defined as 0.

Suppose $\nu(g; G, K)$ is finite. We call an element $a \in G$ which reaches the supremum in either of the conditions stated above, a *minimal root of g in G with respect to K* .

Definition 4.2. Let K be a normal subgroup in G and $S = \{s_1, \dots, s_m\} \subseteq G$. We define $relsize(S; G, K)$ the *relative size of S in G with respect to K* to be

$$relsize(S; G, K) = \sum_{i=1}^m \nu(s_i; G, K)^{-1}.$$

Let M be a finitely generated normal subgroup of G . We define $relsize(M; G, K)$ the *relative size of M in G with respect to K* to be

$$relsize(M; G, K) = \inf_{\substack{S \subseteq G \\ |S| < \infty \\ \langle\langle S \rangle\rangle^G = M}} \{relsize(S; G, K)\}.$$

Remark 4.3. Note that Definition 3.1 may be restated in the following way. Let G and Q be as in Definition 3.1. Denote by \overline{R}_G the inverse image of R_G under $\varphi : F_n \rightarrow G$. Since $R \leq \overline{R}_G$, then

$$\begin{aligned} rdef(Q) &= n - relsize(R; F_n, \overline{R}_G), \\ rdef(G) &= \sup_{\langle X|R \rangle \cong G} \{ |X| - relsize(R; F_n, \overline{R}_G) \}, \end{aligned}$$

where F_n is freely generated by X and the supremum is taken over all finite presentations of G . Note that for $\omega \in F_n$ and N a normal subgroup in F_n , $\nu(\omega; F_n, N)$ is always finite.

Lemma 4.4. *Let G be a group, H a finite index normal subgroup in G , K a normal subgroup in G and g an element in H . If $\nu(g; G, K)$ is infinite, then $\nu(g; H, H \cap K)$ is infinite too.*

Proof. First suppose $g \in K$. Denote by ψ the canonical surjective homomorphism from G onto G/K . If $\nu(g; G, K)$ is infinite, then for all $n \in \mathbb{N}$ there is an element $a \in G$ such that $o(\psi(a), G/K) = m$, where $m \geq n$.

Let $s = |G : H|$ and suppose $\nu(g; H, H \cap K) = C$. Consider $s(C + 1) = n$. Since, $\nu(g; G, K)$ is infinite, there is $a \in G$ such that $o(\psi(a), G/K) = m \geq s(C + 1)$. As $s = |G : H|$, then $a^s \in H$. Since $o(\psi(a), G/K) = m \geq s(C + 1)$, then $o(\psi(a^s), H/K) \geq C + 1$ which contradicts the assumption that $\nu(g; H, H \cap K) = C$.

If $g \notin K$, then instead of considering $o(\psi(a), G/K)$, we consider $n \in \mathbb{N}$, such that $a^n = g$. In this case, the arguments above work equally well. \square

Equivalently, this lemma says that if the relative size of g in G with respect to K is zero, then its relative size in H with respect to $H \cap K$ is also zero.

Lemma 4.5. *Let G be a group. Consider H a finite index normal subgroup of G and K a normal subgroup of G . Let g be an element of H and suppose g has a minimal root a in G with respect to K . Denote by ψ the surjective homomorphism from G to G/H . Then*

$$\nu(g; H, K \cap H) \geq \frac{\nu(g; G, K)}{o(\psi(a), G/H)}.$$

Proof. Let $l = o(\psi(a), G/H)$. First assume $g \notin K$ and $m = \nu(g; G, K)$. In this case $a^m = g$. As $g \in H$ and $a^l \in H$, then l divides m . Since $(a^l)^{m/l} = a^m = g$, then $\nu(g; H, K \cap H) \geq m/l$.

Now suppose $g \in K$. Let $m = \nu(g; G, K)$. By definition of $\nu(g; G, K)$, $a^m \in K$ and $a^j \notin K$ for all $j < m$. If $l = o(\psi(a), G/H)$, then $a^l \in H$. Therefore, if n is the minimum common multiple of l and m , then $a^n \in H \cap K$. Since n is a multiple of l , then l divides n and hence $a^n = (a^l)^{n/l}$.

The element a^l belongs to H . Denote it by b . Then $b^{n/l} \in K$ and $b^i \notin K$ for all $i < n/l$. Since g and b are elements in H , then $\nu(g; H, K \cap H) \geq n/l \geq m/l$, where the last inequality follows from the fact that n is a multiple of m . \square

Theorem 4.6. *Let M and K be normal subgroups of G , where M is finitely generated as a normal subgroup of G . If H is a finite index normal subgroup of G which contains M , then*

$$\text{relsize}(M; H, K \cap H) \leq |G : H| \text{relsize}(M; G, K).$$

Proof. Let $S = \{s_1, \dots, s_d\} \subseteq G$ generate M as a normal subgroup in G . First suppose that the relative size of s_i in G with respect to K is greater than zero for all i , $1 \leq i \leq d$. Hence, consider the set $A = \{a_1, \dots, a_d\}$, where a_i is a minimal root of s_i in G with respect to K . Denote the centraliser of s_i in G by $C_G(s_i)$. Let $k_i = |C_G(s_i) : C_H(s_i)|$ and $l_i = o(\psi(a_i), G/H)$, where ψ is the canonical surjective map from G to G/H .

Let $n_i \in \mathbb{N}$ be such that $a_i^{n_i} = s_i$, where $s_i \in H$. Then for all $t \in \mathbb{Z}$, a_i^t commutes with s_i and hence the subgroup in G generated by a_i is in $C_G(s_i)$ for all i , $1 \leq i \leq d$. Moreover, since $l_i = o(\psi(a_i), G/H)$, then $a_i^{l_i} \in H \cap C_G(s_i)$ and $a_i^r \notin C_H(s_i)$ for all r , $1 \leq r < l_i$. This means that if r and r' are integers such that $1 \leq r, r' < l_i$, with $r \neq r'$, then a_i^r and $a_i^{r'}$ cannot be in the same coset of $C_H(s_i)$ in $C_G(s_i)$. Therefore, for each i , $1 \leq i \leq d$, there are at most l_i distinct cosets of $C_H(s_i)$ in $C_G(s_i)$ which contain an element of the form a_i^t , where $1 \leq t \leq l_i$. Hence, $l_i \leq k_i$, for all i , $1 \leq i \leq d$.

Obtain, as in Lemma 3.3, a set $S' = \{s_{ij}\}$ of generators for M as a normal subgroup of H , where $i = 1, \dots, d$. Note that for each i the index j ranges from 1 to m/k_i , where $m = |G : H|$.

For each i , the set $\{s_{ij}\}$ is obtained by conjugating s_i by the elements in a transversal of $C_G(s_i)$ in G . Therefore, for each i , s_{ij} and $s_{ij'}$ are conjugate to one another in G . Since both K and H are normal in G , the order of the image of a_i in G/H and the order of the image of a_i in G/K are invariant under conjugation. Moreover, if a_i is a minimal root of s_i with respect to K and b is an element in a transversal of $C_G(s_i)$ in G , then $ba_i b^{-1}$ is a minimal root of $bs_i b^{-1}$ with respect to K . Therefore, for each i , $\nu(s_{ij}; H, K \cap H) = \nu(s_{ij'}; H, K \cap H)$ for all j and j' such that $1 \leq j, j' \leq m/k_i$. From this and Lemma 4.5 the following holds

$$\begin{aligned} \text{relsize}(S'; H, K \cap H) &= \sum_{i=1}^d \sum_{j=1}^{m/k_i} \nu(s_{ij}; H, K \cap H)^{-1} = \sum_{i=1}^d \frac{|G : H|}{k_i} \nu(s_i; H, K \cap H)^{-1} \leq \\ &\sum_{i=1}^d \frac{|G : H|}{k_i} l_i \nu(s_i; G, K)^{-1} \leq |G : H| \sum_{i=1}^d \nu(s_i; G, K)^{-1} = \\ &|G : H| \text{relsize}(S; G, K). \end{aligned} \tag{24}$$

Hence, if S generates M as a normal subgroup of G , then S' generates M as a normal subgroup of H , and $\text{relsize}(S'; H, K \cap H) \leq |G : H| \text{relsize}(S; G, K)$. Therefore,

$$\text{relsize}(M; H, K \cap H) \leq \text{relsize}(S'; H, K \cap H) \leq |G : H| \text{relsize}(S; G, K),$$

for all finite generating sets S . The result is then obtained by considering the infimum over all finite generating sets S on the right hand side of the previous inequalities.

Now let us consider the situation when s_i has zero relative size in G with respect to K for some i , $1 \leq i \leq d$. By Lemma 4.4 it has zero relative size in H with respect

to $H \cap K$. Moreover, as H and K are normal in G , and order is preserved under conjugation, then any conjugate of s_i by an element in G also has zero relative size in G and H with respect to K . Therefore, in eq. (24), if s_i has zero relative size in H with respect to K , then s_{ij} , where $1 \leq j \leq m/k_i$, also has zero relative size in H with respect to $H \cap K$, and hence they do not need to be considered in the computation. \square

We now prove that residual deficiency minus one is supermultiplicative.

Theorem 4.7. *Let G be a finitely presented group and H a finite index normal subgroup of G . Then*

$$rdef(G) - 1 \leq \frac{rdef(H) - 1}{|G : H|}. \quad (25)$$

Proof. Let $P = \langle X | T \rangle$ be a finite presentation for G , where $n = |X|$, and let $Q = \langle Y | S \rangle$ be the presentation for H obtained by applying the Reidemeister-Schreier rewriting process to P . Suppose Y freely generates F_k . Note $k = |G : H|(n - 1) + 1$. By eq. (24) in Theorem 4.6

$$relsize(S; F_k, \overline{R}_G) \leq |G : H| relsize(T; F_n, \overline{R}_G).$$

Note that $\overline{R}_G \leq F_k$ in the previous inequality. Therefore,

$$\begin{aligned} |G : H|(rdef(P) - 1) &= |G : H|(n - relsize(T; F_n, \overline{R}_G) - 1) = \\ |G : H|(n - 1) - |G : H| relsize(T; F_n, \overline{R}_G) &\leq |G : H|(n - 1) - relsize(S; F_k, \overline{R}_G) = \\ |G : H|(n - 1) + 1 - relsize(S; F_k, \overline{R}_G) - 1 &= rdef(Q) - 1 \leq rdef(H) - 1. \end{aligned}$$

The result follows by taking the supremum over all finite presentations P of G on the left hand side of the previous inequalities. \square

By Theorem 4.7 the residual deficiency of a group is supermultiplicative on finite index normal subgroups. Therefore, following the spirit in Section 1, we define the *residual deficiency gradient* of G by

$$RDG(G) = \sup_{\substack{H \trianglelefteq G \\ f}} \left\{ \frac{rdef(H) - 1}{|G : H|} \right\},$$

which by Proposition 2.2 gives a multiplicative invariant and hence a generalised Euler characteristic.

However, by Corollary 3.5, the residual deficiency gradient is the same as the deficiency gradient. One inequality is straightforward; it suffices to notice that for every finitely presented group G , $def(G) \leq rdef(G)$. This then implies that $DG(G) \leq RDG(G)$. By Corollary 3.5, $rdef(G) - 1 \leq DG(G)$. Therefore, if N is a finite index normal subgroup of G , then

$$\frac{rdef(N) - 1}{|G : N|} \leq \frac{DG(N)}{|G : N|} = DG(G).$$

Taking the supremum on the left hand side, over all finite index normal subgroups of G , gives $RDG(G) \leq DG(G)$.

5 The residual deficiency of quotients

The following two results give lower bounds for the residual deficiency of quotients of a finitely presented group G .

Proposition 5.1. *Let G be a finitely presented group with non-trivial finite residual R_G . Say s_1, \dots, s_m are non-trivial elements in G contained in R_G . Denote by H the normal subgroup in G generated by $S = \{s_1, \dots, s_m\}$. Then*

$$rdef(G/H) \geq rdef(G) - relsize(S; G, R_G)$$

Proof. Let $P = \langle X | R \rangle$ be a finite presentation for G , where X freely generates F_d , the non-abelian free group of rank d . First, separate S into two disjoint subsets S_1 and S_2 , where the elements in $S_1 = \{s_1, \dots, s_k\}$ have relative size in G with respect to R_G greater than zero, and the ones in $S_2 = \{s_{k+1}, \dots, s_m\}$ do not. Let a_1, \dots, a_k be minimal roots in G with respect to R_G for s_1, \dots, s_k , respectively. Moreover, let $q_1, \dots, q_k \in \mathbb{N}$ be such that $a_i^{q_i} = s_i$, for $1 \leq i \leq k$.

Consider $w_1, \dots, w_k \in F_d$ such that $\varphi(w_1) = a_1, \dots, \varphi(w_k) = a_k$, for all $1 \leq i \leq k$, where φ is the canonical map going from F_d to G . If $w_i = u_i^r$, for $r > 1$, then $\varphi(u_i)^r = \varphi(u_i^r) = \varphi(w_i) = a_i$, which would imply that a_i is not a minimal root for s_i . Therefore, w_i cannot be expressed as a proper power of any other element in F_d . Hence, if $W_1 = \{w_1^{q_1}, \dots, w_k^{q_k}\}$, then

$$relsize(S_1; G, R_G) = relsize(W_1; F_d, \overline{R_G}).$$

If Q is a finite presentation for G/H , then by the definition of residual deficiency, we have $rdef(Q) \leq rdef(G/H)$. We claim that for every $\varepsilon > 0$, there is a finite presentation Q_ε for G/H , such that

$$rdef(Q_\varepsilon) + \varepsilon > rdef(P) - relsize(S; G, R_G).$$

This would then imply

$$rdef(G/H) + \varepsilon > rdef(P) - relsize(S; G, R_G),$$

for all $\varepsilon > 0$, which means

$$rdef(G/H) \geq rdef(P) - relsize(S; G, R_G),$$

for all finite presentations P of G . The result would then be obtained by taking the supremum over all finite presentations for G .

Let $a_{k+1}, \dots, a_m \in G$ be such that $o(\psi(a_j), G/R_G) > (m-k)\varepsilon^{-1}$, for $k+1 \leq j \leq m$, where $\psi : G \rightarrow G/R_G$. Such a_j exist since s_{k+1}, \dots, s_m have zero relative size in G with respect to R_G .

Suppose $a_j^{q_j} = s_j$, for $k+1 \leq j \leq m$. Let $W_2 = \{w_{k+1}^{q_{k+1}}, \dots, w_m^{q_m}\}$ be a subset of F_d such that $\varphi(w_j) = a_j$. Then $\varphi(w_j^{q_j}) = \varphi(w_j)^{q_j} = s_j$, for $k+1 \leq j \leq m$. Hence, $Q_\varepsilon = \langle X | R, W_1, W_2 \rangle$ is a finite presentation for G/H .

First note that

$$rdef(Q_\varepsilon) = rdef(P) - relsize(W_1 \cup W_2; F_d, \overline{R}_G).$$

Second, that

$$relsize(W_1 \cup W_2; F_d, \overline{R}_G) = relsize(W_1; F_d, \overline{R}_G) + relsize(W_2; F_d, \overline{R}_G).$$

We already have that $relsize(W_1; F_d, \overline{R}_G) = relsize(S_1; G, R_G)$. Moreover, given how we chose a_j for $k+1 \leq j \leq m$, we have

$$relsize(W_2; F_d, \overline{R}_G) \leq \sum_{j=k+1}^m o(\psi(a_j), G/R_G)^{-1} < \sum_{j=k+1}^m \frac{\varepsilon}{m-k} = \varepsilon.$$

Therefore,

$$\begin{aligned} rdef(Q_\varepsilon) + \varepsilon &= rdef(P) + \varepsilon - relsize(S_1; G, R_G) - relsize(W_2; F_d, \overline{R}_G) > \\ &= rdef(P) + \varepsilon - relsize(S_1; G, R_G) - \varepsilon. \end{aligned}$$

Since

$$relsize(S; G, R_G) = relsize(S_1, G, R_G) + relsize(S_2, G, R_G),$$

and $relsize(S_2, G, R_G) = 0$, we obtain

$$rdef(Q_\varepsilon) + \varepsilon > rdef(P) - relsize(S; G, R_G).$$

□

Theorem 5.2. *Let G be a finitely presented group. Let g_1, \dots, g_m be non-trivial elements in G with infinite order in G/R_G . Denote by H_n the normal subgroup in G generated by g_1^n, \dots, g_m^n . Then given $\varepsilon > 0$, there exists a natural number k , such that for all $q \in \mathbb{Z} \setminus \{0\}$, we have*

$$rdef(G/H_{kq}) \geq rdef(G) - \varepsilon.$$

Proof. Let $P = \langle X | R \rangle$ be a finite presentation for G , where X generates F_d the non-abelian free group of rank d . Let $R = \{u_1^{a_1}, \dots, u_r^{a_r}\}$, where u_i cannot be expressed as a proper power of any other element in F_d , for all i , $1 \leq i \leq r$. Denote by l_i the order of u_i in F_d/\overline{R}_G .

As F_d/\overline{R}_G is residually finite, there is a finite index normal subgroup K of F_d , which contains \overline{R}_G , such that the order of u_i in F_d/K is l_i , for all i , $1 \leq i \leq r$.

Let T be a positive natural number. Denote by w_j an element in F_d which corresponds to g_j under φ , the canonical homomorphism from F_d to G . As the order of g_j in G/R_G is infinite, so is the order of w_j in F_d/\overline{R}_G . Therefore, there is a finite index normal subgroup L_j of F_d , which contains \overline{R}_G , such that w_j^n is not in L_j for all $n \leq T$. Note that this can be done for all j , $1 \leq j \leq m$.

Take M the intersection of K and $\bigcap_{j=1}^m L_j$. The subgroup M is a finite index normal subgroup of F_d and contains \overline{R}_G . Also, the order of u_i in F_d/M is l_i , for all

$i, 1 \leq i \leq r$. Consider k_j the order of w_j in F_d/M . Then, for every multiple q of k_j , $w_j^q \in M$. Therefore, if k is the minimum common multiple of k_1, \dots, k_m , then $w_j^k \in M$ for all $j, 1 \leq j \leq m$.

Consider $Q = \langle X \mid R, w_1^{kq}, \dots, w_m^{kq} \rangle$, where $q \in \mathbb{Z} \setminus \{0\}$. The presentation Q is a finite presentation for the group G/H_{kq} . If $W = \{w_1^{kq}, \dots, w_m^{kq}\}$, then since $T \leq k_j$ for all $j, 1 \leq j \leq m$,

$$\text{relsize}(W; F_d, \overline{R}_{G/H_{kq}}) \leq \sum_{j=1}^m \frac{1}{k_j} \leq \sum_{j=1}^m \frac{1}{T}.$$

As $\text{rdef}(Q) \geq \text{rdef}(P) - \text{relsize}(W; F_d, \overline{R}_{G/H_{kq}})$, then

$$\text{rdef}(G/H_{kq}) \geq \text{rdef}(Q) \geq \text{rdef}(P) - \sum_{j=1}^m \frac{1}{T} = \text{rdef}(P) - \frac{m}{T}.$$

The natural number T was chosen arbitrarily. Therefore, take T such that $m/T < \varepsilon$. Then $\text{rdef}(P) - m/T > \text{rdef}(P) - \varepsilon$, and hence

$$\text{rdef}(G/H_{kq}) > \text{rdef}(P) - \varepsilon.$$

The result follows by taking the supremum over all finite presentations P of G . \square

Corollary 5.3. *Let G be a finitely presented residually finite group. Let g_1, \dots, g_m be non-trivial elements in G . Denote by H_k the normal group in G generated by g_1^k, \dots, g_m^k . If G has residual deficiency greater than one, then G/H_k has residual deficiency greater than one for an infinite collection of natural numbers k .*

Proof. Let g_1, \dots, g_j have finite order in G while g_{j+1}, \dots, g_m infinite order. Denote by l_1, \dots, l_j the orders of g_1, \dots, g_j , respectively. Let l be the minimum common multiple of l_1, \dots, l_j . Therefore, g_1^l, \dots, g_j^l are all trivial in G .

Since g_{j+1}, \dots, g_m have infinite order in G , then by Theorem 5.2 there is a k such that $G/\langle\langle g_{j+1}^{kq}, \dots, g_m^{kq} \rangle\rangle$ has residual deficiency greater than one for all naturals $q \in \mathbb{N}$.

Take k to be a multiple of l . Note that as g_1^k, \dots, g_j^k are trivial in G , then as normal subgroups in G we have $\langle\langle g_{j+1}^{kq}, \dots, g_m^{kq} \rangle\rangle = \langle\langle g_1^{kq}, \dots, g_m^{kq} \rangle\rangle$. The result then follows. \square

If the residually finite group is a non-abelian free group of finite rank, then the following lemma helps us say something more.

Lemma 5.4. ([30], Lemma 2.4)

For any finite collection of non-trivial elements g_1, \dots, g_m of a non-abelian free group F and any number $k \in \mathbb{N}$, there exists $K \in \mathbb{N}$ with the following property. For every $q \geq K$, there is a finite index normal subgroup $N_q \triangleleft F$ such that for all $1 \leq i \leq m$, $g_i^s \notin N_q$ whenever $1 \leq s \leq k$, but $g_i^q \in N_q$.

Consider $k = m$. By Lemma 5.4, there is an $M \in \mathbb{N}$ such that for all $q \geq M$ there is a finite index normal subgroup N_q in F such that for all i , $1 \leq i \leq m$, $g_i^s \notin N_q$ whenever $1 \leq s \leq m$, but $g_i^q \in N_q$. Consider F_d the non-abelian free group of rank d . Then, the residual deficiency of $F_d / \langle\langle g_1^q, \dots, g_m^q \rangle\rangle$ is greater than or equal to $d - \sum_{i=1}^m 1/(m+1)$, which is greater than one. Hence,

Proposition 5.5. *Let F_d be a non-abelian free group of rank $d \geq 2$, and let g_1, \dots, g_k be arbitrary elements of F_d . Then $G = F_d / \langle\langle g_1^q, \dots, g_m^q \rangle\rangle$ has residual deficiency greater than one for all but finitely many $q \in \mathbb{N}$.*

The authors of [30] used Lemma 5.4 to prove that for all but finitely many $q \in \mathbb{N}$, the group $G = F_r / \langle\langle g_1^q, \dots, g_m^q \rangle\rangle$ has a finite index normal subgroup with deficiency greater than one. Example 3.7 shows that having a finite index normal subgroup with deficiency greater than one does not imply the group has residual deficiency greater than one. Therefore Proposition 5.5 says something stronger about $G = F_r / \langle\langle g_1^q, \dots, g_m^q \rangle\rangle$.

Proposition 5.1 and Theorem 5.2 give lower bounds for the residual deficiency of a quotient G/M , where M is finitely generated. In some cases, the lower bounds are such that if the residual deficiency of G is greater than one, then the residual deficiency of G/M is also greater than one. The following discussion examines a case when the residual deficiency of G/M is not greater than one. The aim is to find conditions under which G/M has a finite index subgroup with residual deficiency greater than one.

Theorem 4.6 says that

$$\text{relsize}(M; H, K \cap H) \leq |G : H| \text{relsize}(M; G, K),$$

where G is a finitely presented group, H a finite index normal subgroup of G , K a normal subgroup of G and M a finitely generated normal subgroup of G contained in H . The proof in Theorem 4.6 first shows that if $S = \{s_1, \dots, s_d\}$ is a generating set for M as a normal subgroup of G , then

$$\text{relsize}(S'; H, K \cap H) \leq |G : H| \text{relsize}(S; G, K), \quad (26)$$

where S' is a generating set for M as a normal subgroup in H , as obtained in Lemma 3.3. However, from eq. (24), the previous inequality is strict when either $\nu(s_i; H, K \cap H)^{-1} < l_i \nu(s_i; G, K)^{-1}$, or $l_i/k_i < 1$, for some i , $1 \leq i \leq d$. Finding conditions when $\nu(s_i; H, K \cap H)^{-1} < l_i \nu(s_i; G, K)^{-1}$ has proved to be difficult. However, there are conditions that imply $l_i/k_i < 1$.

Before specifying these conditions, let us examine the utility of having a strict inequality in eq. (26).

Suppose G is a finitely presented non-residually finite group. Let M be a finitely generated normal subgroup of G , contained in R_G , normally generated in G by $S = \{s_1, \dots, s_d\}$. Moreover, assume $\text{rdef}(G) - \text{relsize}(S; G, R_G) \geq 1$. It is worth noting that by Proposition 5.1

$$1 \leq \text{rdef}(G) - \text{relsize}(S; G, R_G) \leq \text{rdef}(G/M). \quad (27)$$

The residual deficiency of G/M may be greater than one. However, if there is no evident way of verifying it and $l_i/k_i < 1$, then eq. (25) and a strict inequality in eq. (26) give

$$0 \leq |G : H|(rdef(G) - relsize(S; G, R_G) - 1) < rdef(H) - relsize(S'; H, R_G) - 1.$$

Hence,

$$1 < rdef(H) - relsize(S'; H, R_G) \leq rdef(H/M),$$

where the last inequality is a consequence of Proposition 5.1.

Therefore, if we have a strict inequality in eq. (26), the subgroup H/M , which is a finite index normal subgroup of G/M , has residual deficiency greater than one. In particular, all properties implied by having residual deficiency greater than one, which are inherited under finite index supergroups, are enjoyed by G/M .

The following proposition gives conditions under which eq. (26) is a strict inequality.

Proposition 5.6. *Let G be a non-residually finite, finitely presented group, such that $rdef(G) = n > 1$, where $n \in \mathbb{N}$. Let $W = \{w_1, \dots, w_{n-1}\}$ be a set of non-trivial elements in R_G . Suppose w_i , for some i , $1 \leq i \leq n-1$, is such that there is a non-trivial element $a \in C_G(w_i) - R_G$. Then, either $rdef(G/\langle\langle W \rangle\rangle) > 1$, or $G/\langle\langle W \rangle\rangle$ has a finite index subgroup N such that $rdef(N) > 1$.*

Proof. By Proposition 5.1

$$rdef(G/\langle\langle W \rangle\rangle) \geq rdef(G) - relsize(W; G, R_G).$$

Since W has $n-1$ elements, then $relsize(W; G, R_G) \leq n-1$. If $relsize(W; G, R_G) < n-1$, then $rdef(G/\langle\langle W \rangle\rangle) > 1$.

Suppose $relsize(W; G, R_G) = n-1$. First, this means that

$$rdef(G/\langle\langle W \rangle\rangle) \geq rdef(G) - relsize(W; G, R_G) = 1.$$

Second, that $\nu(w_i; G, R_G) = 1$, for all i , $1 \leq i \leq n-1$. This implies that w_i has no minimal root in G with respect to R_G , with order greater than one in G/R_G . In terms of the notation in Theorem 4.6, if H is a finite index normal subgroup of G , then $l_i = 1$ for all i , $1 \leq i \leq n-1$.

Consider a the non-trivial element in $C_G(w_i) - R_G$. Since G/R_G is residually finite, then there is a finite index normal subgroup H in G , such that $a \notin H$. Therefore, $k_i = |C_G(w_i) : C_H(w_i)| > 1$. Since $l_i = 1$ for all finite index normal subgroups H , then $l_i/k_i = 1/k_i < 1$. Then, by the arguments preceding Proposition 5.6, $H/\langle\langle W \rangle\rangle =: N$, is a finite index normal subgroup of $G/\langle\langle W \rangle\rangle$ which has residual deficiency greater than one. \square

Example 5.7. In [37], D. Toledo proved the existence of short exact sequences

$$1 \longrightarrow K \longrightarrow \Phi \longrightarrow \Gamma \longrightarrow 1,$$

with the following properties. The group Γ is residually finite and infinite. The group Φ is finitely presented (the fundamental group of a smooth projective variety) and not

residually finite. The injection $K \longrightarrow \Phi$ is central. That is, K , as a subgroup of Φ , is contained in $Z(\Phi)$, the centre of Φ . Finally, as Φ/K is isomorphic to Γ and Γ is residually finite, then R_Φ is contained in K .

In order to apply Proposition 5.6, we need $rdef(\Phi) \geq 2$. We do not know what is the residual deficiency of Φ , so take $k \in \mathbb{N}$ such that $rdef(\Phi) + k \geq 2$ and consider $G = \Phi * F_k$. Therefore, $rdef(G) = rdef(\Phi * F_k) \geq rdef(\Phi) + k \geq 2$. Moreover, since $\Phi \leq G$, then G is not residually finite.

Note that $R_G \cap \Phi = R_\Phi$. Therefore, if $\phi \in \Phi - R_\Phi$ then $\phi \notin R_G$. As $\Gamma \cong \Phi/K$ is non-trivial and $R_\Phi \leq K$, then such ϕ exists.

Consider $W = \{w_1, \dots, w_n\} \in R_\Phi$, where n is such that $rdef(\Phi) + k - n = 1$. As R_Φ is contained in $Z(\Phi)$, then w_i commutes with all elements of Φ , for all i , $1 \leq i \leq n$. Consider $\{t_1, \dots, t_k\}$ a free set of generators for F_k . Let t be any of them. Then tw_it^{-1} commutes with all elements in $t\Phi t^{-1}$, for all i , $1 \leq i \leq n$.

So far we have $tw_it^{-1} \in tR_\Phi t^{-1} \leq tZ(\Phi)t^{-1} \leq t\Phi t^{-1} \leq G$. We also know that $tR_\Phi t^{-1} \leq R_G$ and that if $\phi \in \Phi - R_\Phi$, then not only $\phi \notin R_G$ but also $t\phi t^{-1} \notin R_G$ as R_G is normal in G . Since $t\phi t^{-1}$ commutes with tw_it^{-1} for all $\phi \in \Phi - R_\Phi$ and all i , then $t\phi t^{-1} \in C_G(tw_it^{-1}) - R_G$ for all $\phi \in \Phi - R_\Phi$ and any i , $1 \leq i \leq n$.

Let $tWt^{-1} = \{tw_1t^{-1}, \dots, tw_nt^{-1}\}$. We already have that if $\phi \in \Phi - R_\Phi$, then $t\phi t^{-1} \in C_G(tw_it^{-1}) - R_G$. Therefore, by Proposition 5.6, either the residual deficiency of $G/\langle\langle tWt^{-1} \rangle\rangle$ is greater than one, or $G/\langle\langle tWt^{-1} \rangle\rangle$ has a finite index subgroup with residual deficiency greater than one.

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